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Random Choice Solution of Hyperbolic Systems*

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A random choice method for solving nonlinear hyperbolic systems of conservation laws is presented. The method is rooted in Glimm's constructive proof that such systems have solutions. The solution is advanced in time by a sequence of operations which includes the solution of Riemann problems and a sampling procedure. The method can describe a complex pattern of shock wave and slip line interactions without introducing numerical viscosity and without a special handling of discontinuities. Examples are given of applications to one- and two-dimensional gas flow problems.

OUTLINE OF GOAL AND METHOD

The goal of the present paper is to present a numerical method for solving nonlinear hyperbolic systems, in particular those which arise in gas dynamics. The method is meant to be usable when the solution sought exhibits a complex pattern of shock waves and slip lines, and possibly large energy densities; it is meant to be useful when methods which rely on either an artificial viscosity or on a special treatment of discontinuitites become impractical because they are either too difficult to program or too expensive to run. The particular applications we have in mind are to combustion problems in engines, where a complex wave pattern is known to exist [11], and where the use of finite differences is undesirable because numerical viscosity cannot be allowed and because there arise stability problems which are hard to overcome [3].

The method of computation evolved from Glimm's constructive existence proof [5], which will now be described. Consider the strictly nonlinear system of equations

$$\mathbf{u}_t = (\mathbf{f}(\mathbf{u}))_x , \tag{1}$$

where $\mathbf{u}$ is a solution vector and the subscripts denote differentiation. (For a definition of strict nonlinearity, see [8].) Let the initial data $\mathbf{u}(x, 0)$ be close to constant (the specific assumptions are spelled out in [5]). The time t is divided into

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intervals of length k. Let h be a spatial increment. The solution is to be evaluated at time $t = nk$, n integer, at the points ih, $i = 0, \pm 1, \ldots$, and at time $(n + \frac{1}{2})k$ at the points $(i + \frac{1}{2})h$. Let $\mathbf{u}_i^n$ approximate $\mathbf{u}(ih, nk)$, and let $\mathbf{u}_{i+1/2}^{n+1/2}$ approximate $\mathbf{u}((i + \frac{1}{2})h, (n + \frac{1}{2})k)$. To find $\mathbf{u}_{i+1/2}^{n+1/2}$ given $\mathbf{u}_i^n$, $\mathbf{u}_{i+1}^n$ (and thus define the algorithm) one begins by considering an initial value problem for Eqs. (1) with the discontinuous initial data

$$\mathbf{u}(x, 0) = \mathbf{u}_{i+1}^n \qquad \text{for} \quad x \geqslant 0,$$

$$\mathbf{u}(x, 0) = \mathbf{u}_i^n \qquad \text{for} \quad x < 0.$$

(Such a problem is called a Riemann problem.) Let $\mathbf{v}(x, t)$ denote the solution of this Riemann problem, and let θ_i be a value of a random variable θ equidistributed in $[-\frac{1}{2}, \frac{1}{2}]$; let P_i be the point $(\theta_i h, k/2)$, and let $\bar{\mathbf{u}} = \mathbf{v}(P_i) = \mathbf{v}(\theta_i h, k/2)$ be the value of the solution of the Riemann problem at P_i. Set

$$\mathbf{u}_{i+1/2}^{n+1/2} = \bar{\mathbf{u}}.$$

A similar construction allows one to proceed from $\mathbf{u}_{i+1/2}^{n+1/2}$ to $\mathbf{u}_i^{n+1}$. In [5], Glimm showed that under certain conditions, $\mathbf{u}$ converges to a weak solution of (1). In the next four sections, it will be shown how this construction can be made into an efficient numerical tool.

In the first of these sections, the algorithm is applied to the solution of the equations of gas dynamics in one space dimension. The treatment of boundary conditions, the transport of passive quantities, and, most importantly, the sampling procedure for θ, are described. An appropriate choice of sampling procedure is crucial to the success of the method.

In the second of these sections, the method of solution of the Riemann problems is described, following in the main the ideas of Godunov [6].

In the third section, the method is generalized to multidimensional problems. A section is then devoted to examples, and conclusions are given at the end.

An earlier attempt to program Glimm's method is described in [10]. An interesting discussion is given in [9].

Implementation of Glimm's Method

The equations of gas dynamics in one space dimension can be written in the (conservation) form

$$\rho_t + (\rho u)_x = 0, \tag{2a}$$

$$(\rho u)_t + (\rho u^2 + p)_x = 0, \tag{2b}$$

$$e_t + ((e + p)\, u)_x = 0, \tag{2c}$$

where the subscripts denote differentiation, ρ is the density of the gas, u is the velocity, ρu is the momentum, e is the energy per unit volume, and p is the pressure. We have

$$e = \rho\epsilon + \tfrac{1}{2}\rho u^2, \tag{2d}$$

where ϵ is the internal energy per unit mass; furthermore, we assume the gas is polytropic, and thus

$$\epsilon = (1/(\gamma - 1))(p/\rho), \tag{2e}$$

where γ is a constant, $\gamma > 1$. Equations (2) are not strictly nonlinear in the sense of [5, 8], and our data will not be close to constant. Thus belief in the convergence of the method becomes at least temporarily the result of computational experience rather than the consequence of a proof.

It will be assumed in this section that a method for solving a Riemann problem and sampling the solution is available; i.e., it is assumed that given a right state S_r with $\rho = \rho_r$, $u = u_r$, $p = p_r$, $e = e_r$ in $x \geqslant 0$, and a left state S_l with $\rho = \rho_l$, $u = u_l$, $p = p_l$, $e = e_l$ in $x < 0$, the solution of Eqs. (2) can be found. This solution will consist of three states: S_r, S_l, a middle state S_* with $u = u_*$, $p = p_*$, separated by waves which may be either shocks or rarefaction waves. A slip line $dx/dt = u_*$ separates the gas initially at $x < 0$ from the gas initially at $x \geqslant 0$. u and p are continuous across the slip line, while ρ in general is not (the discontinuity of ρ would be excluded in a strictly nonlinear system). The slip line divides S_* into two parts with possibly differing values of ρ_* and e_*, but equal constant values of u_* and p_* (see Fig. 1).

Given the solution of the Riemann problem evaluated at the sample point $P = (\theta h, k/2)$, Glimm's construction can be carried out. The choice of θ of course determines the behavior of the solution. Thus, if θ is close to $-\frac{1}{2}$, the values in S_l propagate to $((i + \frac{1}{2})h, (n + \frac{1}{2})k)$, while if θ is close to $\frac{1}{2}$ the values in S_r propagate to their left.

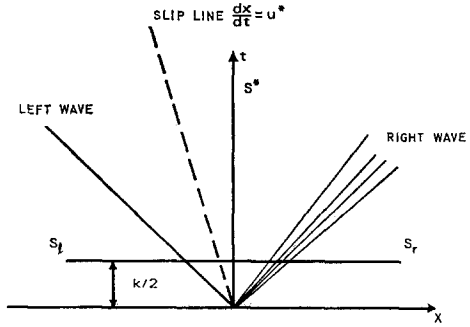

FIG. 1. Solution of Riemann problem.

If $\theta_1, \theta_2, \ldots, \theta_n, \ldots$ are the successive values of θ used in the calculation, it is important that they tend as fast as possible to approximate equipartition on $[-\frac{1}{2}, \frac{1}{2}]$. For more precise requirements, see [5]. It is natural to think of picking a new value of θ for each i and each n. The practical effect of such a choice with finite h is disastrous, except for data very close to a constant. In particular, with such a choice, there is a finite probability that a given state S will propagate to both left and right and create a spurious constant state. The first improvement in the method is a choice of a new θ only once per time level rather than once for each point and each time level. In Table I we give an example of the improvement in

TABLE I

Effect of the Sampling Procedure on the Quality of the Solution[a]

x	Run A	Run B
0.40	−1.000	−1.000
0.46	−1.000	−0.839
0.53	−0.963	−0.668
0.60	−0.963	−0.594
0.66	−0.963	−0.494
0.73	−0.963	−0.402
0.80	0.	−0.382
0.86	0.	0.

[a] Velocity field in a rarefaction wave, $t = 0.40$.

the quality of the solution which results from such a choice: Consider a gas with $\rho = 1$, $p = 1$, $u = 0$, in $0 \leqslant x \leqslant 1$. At time $t = 0$ the left boundary is set into motion with velocity $V = -1$ (we shall describe below how boundary conditions are imposed). The two columns of numbers were obtained by processes which differ only in the choice of θ. In Run A a new θ was chosen for each space interval and each time level, while in Run B a new θ was chosen for each time level only.

The variance of the solution can be further reduced by the following procedure, whose goal is to make the sequence of samples θ_i reach approximate equidistribution over $[-\frac{1}{2}, \frac{1}{2}]$ at a faster rate. Let $m_1, m_2, m_1 < m_2$, be two mutually prime (or even better, prime) integers. Consider the sequence of integers

$$n_0 \text{ given}, \quad n_0 < m_2,$$
$$n_{i+1} = (m_1 + n_i) (\text{mod } m_2).$$

(This mimics on the integers a procedure by which pseudorandom numbers are generated on the computer.) Consider the sequence of samples $\theta_1, \theta_2, \ldots$, of θ, and introduce the modified sequence

$$\theta_i' = ((n_i + \theta_i + \tfrac{1}{2})/m_2) - \tfrac{1}{2}, \qquad i = 1, 2, \ldots.$$

The θ_i' will be used instead of θ_i. To see the advantages of the modified procedure, consider a shock moving at a constant speed U between two constant states, $Uk < h$. The position X of the shock after n half-steps is given by

$$X = h \sum_{i=1}^{n} \eta_i,$$

where the η_i are random variables,

$$\begin{aligned} \eta_i &= \tfrac{1}{2} \qquad \text{if} \quad h\theta_i < Uk/2, \\ &= -\tfrac{1}{2} \qquad \text{if} \quad h\theta_i \geqslant Uk/2. \end{aligned}$$

The standard deviation of X is

$$h(n)^{1/2}\{(\tfrac{1}{2} + U(k/h))(\tfrac{1}{2} - U(k/h)\}^{1/2}.$$

The standard deviation is a measure of the statistical error. Its maximum value is reached when $Uk = 0$, when its value is $h(n)^{1/2}/2$. If the θ_i' are used instead of the θ_i, the maximum standard deviation in X becomes $\frac{1}{2}h(n/m_2)^{1/2}$; furthermore, if the θ_i are used, we have

$$-nh/2 \leqslant X \leqslant nh/2;$$

the extreme values $\pm nh/2$ can be reached (although with a very low probability); on the other hand, if the θ_i' are used, we have

$$m_3/m_2 \leqslant X \leqslant (m_3 + 1)/m_2$$

whenever n is a multiple of m_2, and where m_3 is the integer part of $Unkm_2/2$. When h is finite, m_2 cannot be made too large; otherwise, it introduces a systematical error into the calculation. We usually picked $m_2 = 7$, $m_1 = 3$. The convergence when $h \to 0$ is not affected as long as m_2 remains bounded.

Boundary conditions can be satisfied through the use of symmetry considerations. Consider a boundary point at $x = b$, with the fluid to the left. The boundary conditions are imposed on the grid point closest to b, say $i_0 h$. A fake right state S_r at $(i_0 + \frac{1}{2})\, h$ is created by setting

$$\rho_{i_0+1/2} = \rho_{i_0-1/2}, \tag{3a}$$

$$u_{i_0+1/2} = -u_{i_0-1/2}, \tag{3b}$$

$$p_{i_0+1/2} = p_{i_0-1/2}. \tag{3c}$$

One then samples that part of the resulting Riemann problem which lies to the left of the slip line. If the variance reduction technique described above is used, the θ used at the boundary must be independent of the θ used in the rest of the calculation, or else there may be a finite probability that waves reflected from the boundary may never actually penetrate the interior of the domain. If an additional point $(i_0 + 1)\,h$ must be added to the grid, the values of ρ, u, p at this point are set equal to the values of the solution at the Riemann problem just described, sampled just to the left of the slip line.

If a passive quantity ψ is transported by the fluid. i.e., if the equation

$$\psi_t + (u\psi)_x = 0 \tag{4}$$

is added to Eqs. (2), it can be readily seen that

$$\begin{aligned} \psi_{i+1/2}^{n+1/2} &= \psi_i^n \qquad \text{if} \quad P = (h, k/2) \text{ is to the left of the slip line,} \\ &= \psi_{i+1}^n \qquad \text{if} \quad P \text{ is to the right of the slip line.} \end{aligned}$$

As a result, the region in which $\psi \neq 0$ remains sharply defined if it is sharply defined initially. Shocks are kept perfectly sharp. If ψ is a step function at $t = 0$, it remains a step function for all t. We express this by the statement that the method has no numerical viscosity.

The accuracy of the method cannot be assessed by usual means. There is no $n > 0$ such that the method provides an exact solution whenever that solution is a polynomial of degree n. For systems of equations with constant coefficients, the method yields a solution which exactly equals the correct solution except for a rigid translation of the coordinate system, by an amount which, with appropriate m_2, is $O(h)$. The method has then first-order accuracy but infinite resolution. In nonlinear systems, and in the presence of boundaries, the situation is more complex, and will be analyzed elsewhere.

The method is found experimentally to be unconditionally stable; in particular, it is stable when the Courant condition is violated. This fact is not paradoxical: since θ is allowed to range only over the interval $[-\frac{1}{2}, \frac{1}{2}]$, all information from outside the domain of dependence of a point is disregarded. Thus if the Courant condition is violated, the problem effectively being solved is modified.

To complete the description of the algorithm, a method for solving Riemann problems must be described. This is the object of the next section.

Solution of a Riemann Problem

The solution of a Riemann problem, needed for the application of the method just described, requires a rather lengthy explanation (and a rather lengthy program),

but, since only one of the possible waves has to be fully computed, it is not uneconomical in terms of computer time.

As already described, we have at $t = 0$ two states, a left state S_l with $\rho = \rho_l$, $u = u_l$, $p = p_l$, and a right state S_r with $\rho = \rho_r$, $u = u_r$, $p = p_r$. We wish to find the solution $\bar{\rho}, \bar{u}, \bar{p}$ at the sample point $P = (\theta h, k/2)$, $-\frac{1}{2} \leqslant \theta \leqslant \frac{1}{2}$. The solution consists of S_r on the right, S_l on the left, and a state S_* in the center. S_* is separated from S_r by a right wave which is either a shock or a rarefaction, and from S_l by a left wave which is also either a shock or a rarefaction (see [4, 5, 8]).

The first step is to evaluate the pressure p_* and the velocity u_* in S_*. This is done by a method due in the main to Godunov ([6]; see also [12]). Define the quantity

$$M_r = (p_r - p_*)/(u_r - u_*). \tag{5}$$

One can show that if the right wave is a shock,

$$M_r = -\rho_r(u_r - U_r) = -\rho_*(u_* - U_r), \tag{6}$$

when ρ_* is the density in the portion of S_* adjoining the right shock and U_r is the velocity of the right shock (see [11]). In any case, one has

$$M_r = (p_r\rho_r)^{1/2}\,\phi(p_*/p_r). \tag{7}$$

where

$$\begin{aligned} \phi(w) &= \left(\frac{\gamma+1}{2}\,w + \frac{\gamma-1}{2}\right)^{1/2} && \text{for} \quad w \geqslant 1, \\ &= \frac{\gamma-1}{2(\gamma)^{1/2}}\,\frac{1-w}{1-w^{(\gamma-1)/2\gamma}} && \text{for} \quad w \leqslant 1, \end{aligned}$$

($\phi(1) = (\gamma)^{1/2}$). Similarly, M_l is defined by

$$M_l = (p_l - p_*)/(u_l - u_*); \tag{8}$$

if the left wave is a shock,

$$M_l = \rho_l(u_l - U_l) = \rho_*(u_* - U_l), \tag{9}$$

where ρ_* is the density to the right of the left shock and U_l is the velocity of the left shock; in any case,

$$M_l = (p_l\rho_l)^{1/2}\,\phi(p_*/p_l), \tag{10}$$

where $\phi(w)$ is defined as in Eqs. (7). From (5) and (8), we have

$$p_* = (u_l - u_r + p_r/M_r + p_l/M_l)/((1/M_r) + (1/M_l)). \tag{11}$$

These considerations lead to the following iteration procedure. Pick a starting value p_*^0 (or values M_r^0, M_1^0), and then compute p_*^{q+1}, M_r^{q+1}, M_l^{q+1}, $q \geqslant 0$, using

$$\tilde{p}^q = (u_1 - u_r + p_r/M_r^q + p_1/M_1^q)/((1/M_r^q) + (1/M_1^q)), \tag{12a}$$

$$p_*^{q+1} = \max(\epsilon_1, \tilde{p}^q), \tag{12b}$$

$$M_r^{q+1} = (p_r\rho_r)^{1/2}\,\phi(p_*^{q+1}/p_r), \tag{12c}$$

$$M_1^{q+1} = (p_1\rho_1)^{1/2}\,\phi(p_*^{q+1}/p_1). \tag{12d}$$

Equation (12b) is needed because there is no guarantee that in the course of iteration $\tilde{p}$ remains $\geqslant 0$. We usually set $\epsilon_1 = 10^{-6}$. The iteration is stopped when

$$\max(|\, M_r^{q+1} - M_r^q \,|, |\, M_1^{q+1} - M_1^q \,|) \leqslant \epsilon_2,$$

(we usually picked $\epsilon_2 = 10^{-6}$); one then sets $M_r = M_r^{q+1}$, $M_1 = M_1^{q+1}$, and $p_* = p_{q+1}^*$.

To start this procedure one needs initial values of either M_r and M_1 or p_*. The starting procedure suggested by Godunov appears to be ineffective, and better results were obtained by setting

$$p_*^0 = (p_r + p_1)/2.$$

We also ensured that the iteration was carried out at least twice, to avoid spurious convergence when $p_r = p_1$.

As noted by Godunov, the iteration may fail to converge in the presence of a strong rarefaction. This problem can be overcome by the following variant of Godunov's procedure. If the iteration has not converged after L iterations (we usually set $L = 20$), Eq. (12b) is replaced by

$$p_*^{q+1} = \alpha \max(\epsilon_1, \tilde{p}^q) + (1 - \alpha)\, p_*^q, \tag{12b}'$$

with $\alpha = \alpha_1 = \frac{1}{2}$. If a further L iteration occurs without convergence, we reset $\alpha_2 = \alpha_1/2$. More generally, the program was written in such a way that if the iteration fails to converge after lL iterations (l integer), α is reset to

$$\alpha = \alpha_l = \alpha_{l-1}/2.$$

In practice, the cases $l > 2$ were never encountered. The number of iterations required oscillated between 2 and 10, except at a very few points.

Once p_*, M_r, M_1 are known, we have

$$u_* = (p_1 - p_r + M_r u_r + M_1 u_1)/(M_r + M_1)$$

from the definitions of M_r and M_1.

The fluid initially at $x \leqslant 0$ is separated from the fluid initially at $x > 0$ by a slip line whose inverse slope is $(dx/dt) = u_*$. There are now four cases to be considered:

(A) The sample point $P = (\theta h, k/2)$ lies to the left of the slip line, i.e., $\theta h \geqslant u_* k/2$, and the right wave is a shock, i.e., $p_* > p_r$;

(B) the sample point P lies to the right of the slip line and the right wave is a rarefaction, i.e., $\theta h \geqslant u_* k/2$ and $p_* \leqslant p_r$;

(C) P lies to the left of the slip line and the left wave is a shock, i.e., $\theta h < u_* k/2$ and $p_* > p_l$; and

(D) P lies to the left of the slip line and the left wave is a rarefaction, i.e., $\theta h < u_* k/2$ and $p_* \leqslant p_l$.

Case A. The velocity U_r of the right shock can be found by using Eq. (6). If the sample point P lies to the right of the shock line $dx/dt = U_r$ we have $\bar{\rho} = \rho_r$, $\bar{u} = u_r$, $\bar{p} = p_r$. If P lies to the left of the shock, $\bar{u} = u_*$, $\bar{p} = p_*$; $\bar{\rho} = \rho_*$ can be found from the second of Eqs. (6).

Case B. Let $c = (\gamma p/\rho)^{1/2}$ be the sound speed. The rarefaction is bounded on the right by the line $dx/dt = u_r + c_r$, $c_r = (\gamma p_r/\rho_r)^{1/2}$, and on the left by $dx/dt = u_* + c_*$, where $c_* = (\gamma p_*/\rho_*)^{1/2}$ can be found by using the constancy of the Riemann invariant:

$$\Gamma_r = 2c_*(\gamma - 1)^{-1} - u_* = 2c_r(\gamma - 1)^{-1} - u_r .$$

If P lies to the right of the rarefaction, $\bar{\rho} = \rho_r$, $\bar{u} = u_r$, $\bar{p} = p_r$; if P lies to the left of the rarefaction, $\bar{\rho} = \rho_*$, $\bar{u} = u_*$, $\bar{p} = p_*$. If P lies inside the rarefaction, we equate the slope of the characteristic $dx/dt = u + c$ to the slope of the line through the origin and P, obtaining

$$\bar{u} + \bar{c} = 2\theta h/k;$$

the constancy of Γ_r and the isentropic law $p\rho^{-\gamma} =$ constant, together with the definition $c = (\gamma p/\rho)^{1/2}$, yield $\bar{\rho}$, $\bar{u}$, and $\bar{p}$.

Cases C and D are mirror images of cases A and B, with M_r , Γ_r replaced by M_l and $\Gamma_l = 2c(\gamma - 1)^{-1} + u$, and will not be described in full.

Multidimensional Problems

We now generalize our version of Glimm's method to problems in two space dimensions. Problems in spaces of greater dimension can presumably be handled

in a similar manner. The equations to be solved can be written in the conservation form (see, e.g., [12]):

$$\rho_t + (\rho u)_x + (\rho v)_y = 0, \tag{13a}$$

$$(\rho u)_t + (\rho u^2 + p)_x + (\rho uv)_y = 0, \tag{13b}$$

$$(\rho v)_t + (\rho uv)_x + (\rho v^2 + p)_y = 0, \tag{13c}$$

$$e_t + ((e + p)u)_x + ((e + p)v)_y = 0, \tag{13d}$$

where $\mathbf{u} = (u, v)$ is the velocity vector, x and y are the spatial coordinates, and e is the energy per unit volume, with

$$e = \rho\epsilon + \tfrac{1}{2}\rho(u^2 + v^2), \tag{13e}$$

where ϵ is the internal energy per unit mass, with the relation

$$\epsilon = (1/(\gamma - 1))(p/\rho) \tag{13f}$$

holding for polytropic gas.

The basic procedure is the use of the Glimm algorithm as a building block in a fractional step method. At each time step four quarter steps of duration $k/2$ are performed; each quarter step is a sweep in either the x or y direction. The equations to be solved in the x sweeps are

$$\rho_t + (\rho u)_x = 0, \tag{14a}$$

$$(\rho u)_t + (\rho u^2 + p)_x = 0, \tag{14b}$$

$$(\rho v)_t + (\rho uv)_x = 0, \tag{14c}$$

$$e_t + ((\rho + p)\, u)_x = 0. \tag{14d}$$

Equation (14c) can be rewritten in the (nonconservation) form

$$v_t + (uv)_x = 0, \tag{14c}'$$

from which it can be seen that in the x sweep v is transported as a passive scalar. Equation (14c) guarantees the conservation in the mean of v; thus Eq. (13e) can be replaced by

$$e = \rho\epsilon + \tfrac{1}{2}\rho u^2 + \text{constant}; \tag{14e}$$

the constant plays no role. Equation (13f) remains valid. Similar equations hold in the y sweeps.

There is no contradiction between the use of a splitting technique and the basic Glimm procedure. At each partial step, the solution vector is approximated by a piecewise constant vector. In the x sweeps the resulting waves in the x direction

are found, and in the y sweeps the waves in the y direction are found. The task at hand is to combine the fractional steps in such a way that in the mean the interaction of the x waves and the y waves is properly accounted for.

This can be readily accomplished as follows: At the beginning of the time step ρ, p, and u are known at the points (ih, jh). After an x sweep, the solution at $((i + \frac{1}{2})h, jh)$ is found (see Fig. 2). The solution is approximated by functions

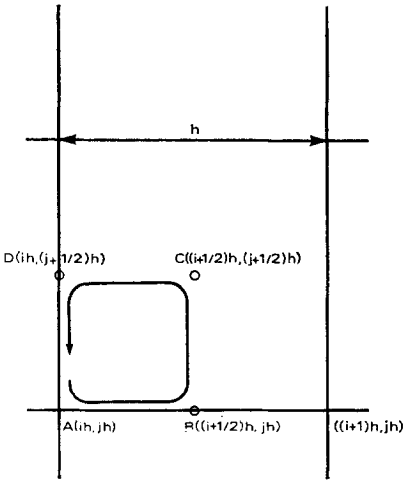

FIG. 2. Grid configuration.

constant on squares centered at these new points, and y waves are sampled, yielding a solution at $((i + \frac{1}{2})h, (j + \frac{1}{2})h)$. An x sweep then leads to $(ih, (j + \frac{1}{2})h)$, and a y sweep to (ih, jh). In Fig. 2 the direction of computation is $ABCDA$. One pseudorandom number is used per quarter step.

Boundary conditions present a challenge to the application of the present method. The basic technique used is reflection. If the boundary is parallel to the mesh, no problems arise; the reflection techniques of the one-dimensional algorithm can be adapted without further ado. If the boundary lies obliquely on the grid, some suitable interpolation procedure must be found. We now describe an interpolation procedure which we found useful when the angle α between the boundary and one of the coordinate axes was small (see Fig. 3). The numerical boundary is made up of points (ih, jh) with i, j integers. Thus, boundary conditions are needed only in the third and fourth quarter steps. Let the boundary be nearly parallel to the x axis. In the third sweep, which is in the y direction, values of $\mathbf{u}$, ρ, and p are needed at points such as R, lying on a vertical line just below the boundary. Values at Q are available, and thus by reflection, values at Q' can be obtained; Q' is symmetrical to Q with respect to the boundary. At Q', ρ and p

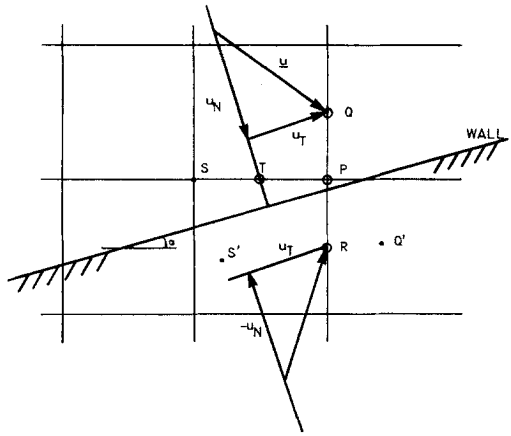

FIG. 3. Boundary conditions.

equal ρ and p at Q, the component of the velocity tangential to the boundary is the same as at Q, and the normal component has changed its sign. Similarly, values of ρ, p, and $\mathbf{u}$ at S' can be obtained by reflecting the values at S. Values at the point R can be obtained by interpolation; we can write approximately

$$\begin{aligned} \rho_R &= (1 - \tan\alpha)\,\rho_{Q'} + (\tan\alpha)\,\rho_{S'}, \\ p_R &= (1 - \tan\alpha)\,p_{Q'} + (\tan\alpha)\,p_{S'}, \\ \mathbf{u}_R &= (1 - \tan\alpha)\,\mathbf{u}_{Q'} + (\tan\alpha)\,\mathbf{u}_{S'}. \end{aligned} \tag{15}$$

In the last sweep, boundary conditions on the right are needed. The boundary is nearly horizontal, and thus, if (ih, j_1h), $((i+1)h, j_2h)$ are two neighboring boundary points, one often has $j_2 = j_1$, and thus the boundary conditions on the right do not have to be specified. If $j_2 = j_1 + 1$, then one sets ρ, p, $\mathbf{u}$ on the right equal to the values derived from (15), where $S = (ih, j_1h)$, and $Q = ((i+1)h, j_2)$, $j_2 = j_1 + 1$.

The method is now completely specified. We leave a discussion of the circumstances under which it is useful to the concluding section.

EXAMPLES

It is easy to identify problems in which the present method provides answers of spectacular quality; for example, in one-dimensional flow, shocks remain perfectly sharp; passive quantities are transported without diffusion in any number of dimensions, and strong shocks present no difficulties. To give a good feeling

for the solutions obtained with our method, we picked a problem which is not tailored to the shape of our method; this problem involves weak shocks, oblique boundaries, discontinuities which form an angle of nearly 45_ϕ with the grid, and a steady state which is very sensitive to small perturbations, of just the kind the algorithm generates in abundance. It is hoped that the successful completion of the calculation will serve as a trustworthy testimonial to the power of the method.

The problem in question involves flow in a constricted channel. It was used as a test problem by Burstein [1] for two-dimensional Lax–Wendroff methods, and by Harten [7] for his artificial compression method, whose purpose is to sharpen shocks and slip lines. A supersonic flow enters a channel with a wedge angle α (see Figs. 4 and 5). In all our calculations, $\alpha = \tan^{-1}(1/5)$. When the Mach number

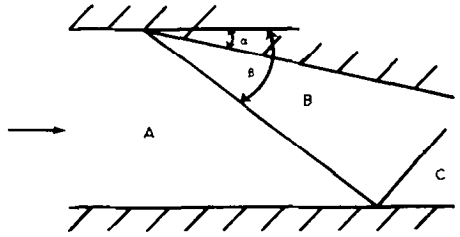

FIG. 4. Regular reflection pattern.

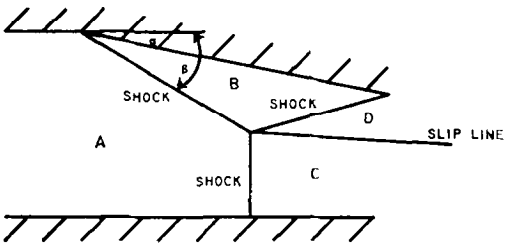

FIG. 5. Mach reflection pattern.

$M = 2$ and $\gamma = 1.4$, a regular reflection occurs at the lower boundary (Fig. 4). When $M = 1.6$ and $\gamma = 1.2$, a Mach reflection occurs (Fig. 5). In all our calculations, the flow was started impulsively, i.e., at $t = 0$ the conditions everywhere equaled the conditions on the left.

In Fig. 6 we present the density field evaluated at time $t = 6.31$, obtained with $h = 1/17$, $k = 0.0147$, $k/h = 0.25$ (the Courant condition is barely satisfied), with 17 nodes in the x direction and 12 nodes in the y direction. The running time was about 12 minutes on a CDC 6400 computer. The left boundary is maintained at the constant state $\rho = 1$, $p = 1$, $v = 0$, $u = 2(\gamma p/\rho)^{1/2} = 2.37$ (Mach number $M = 2$); $\gamma = 1.4$. At the right, the fluid is allowed to flow out freely. A shock is

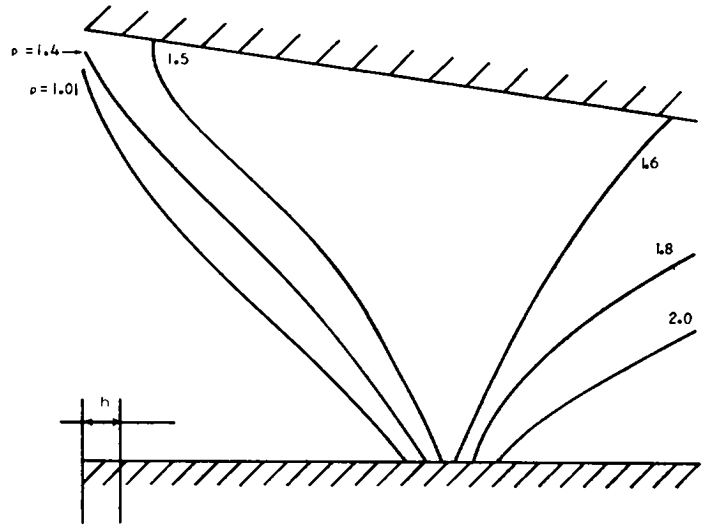

FIG. 6. Density field, Mach number $M = 2$, $\gamma = 1.4$.

induced by the wedge, and is reflected at the lower boundary. The shock angle β (Fig. 4) should be 40.8 degrees; the computed shock angle is indistinguishable from this value; it is of course not computed accurately on such a crude grid. The exact value of ρ in region B is $\rho = 1.52$, and in region C, $\rho = 2.27$ (see [4]). In Table II we give the values of ρ along the lower boundary. The position of the shocks is marked by an arrow.

In Fig. 7 we display the density field obtained with $M = 1.6$, $\gamma = 1.2$. A Mach reflection occurs. In region B the exact value of ρ is 1.48. In region A we have of couse $\rho = 1$. We used $h = 1/14$, $k/h = 0.25$, a grid of 17×12, and ran 184 time steps. Since the upstream flow is subsonic, small perturbations affect the calculation and on the right the steady state has not been reached. With good will, the slip line can be seen.

These results are at least equal in quality to those obtained previously, and much superior to those one can expect from any straightforward first-order method (see [6]). It can be seen that the steady state is never fully achieved. The correct shock transition occurs; it is built into the method, and besides, the method conserves energy, momentum, and mass in the mean. One feature of the results obtained is that they contain fluctuation of a small scale and amplitude. This can be eliminated through the use of a small artificial viscosity (the present method has none), but such a viscosity would destroy one of the most important advantages of the method. Besides, a quick glance at a real flow field will show that small fluctuations hardly detract from the physical meaning of the computed results.

TABLE II

Density at Lower Wall[a]

$t = 5.30$ ($n = 360$)	$t = 6.32$ ($n = 430$)	
1.00	1.00	
1.00	1.00	
1.00	1.00	exact $\rho = 1$
1.01	1.00	
1.02	1.00	
1.08	1.51	
1.26	1.85	exact position of ← triple point
1.58	1.92	
2.04	2.38	
2.41	2.60	exact $\rho = 2.27$
2.22	2.49	
2.24	2.38	

[a] Mach No. $M = 2$, $\gamma = 1.6$, regular reflection, 17×12 grid.

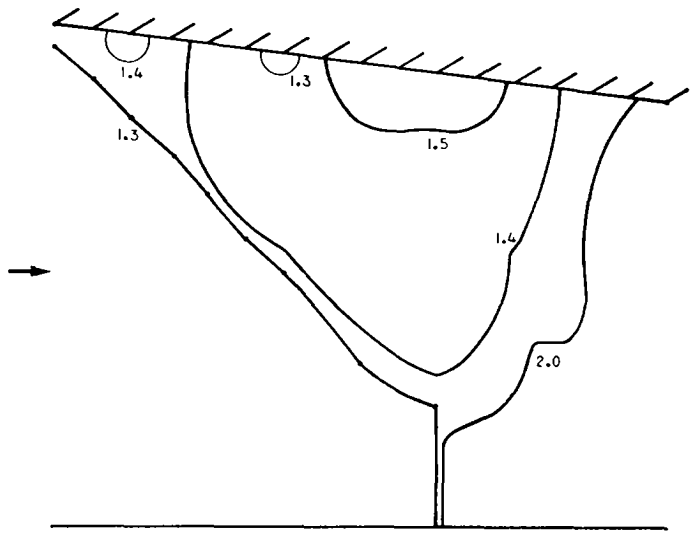

FIG. 7. Density field, $M = 1.6$, $\gamma = 1.2$.

Conclusions

We have developed a random choice method for solving the equation of gas dynamics. This method is obviously not competitive with more classical methods when it is applied to problems whose solutions are smooth, since in such problems greater accuracy can be achieved with much smaller effort. The method is destined for use in problems which involve complex patterns of discontinuity; in such problems, the greater effort required per mesh point is balanced by the economy in representation, which requires fewer points per problem.

The interesting features of this explicit method are its unconditional stability, and its neglect of all characteristic velocities larger than h/k. Those are features one desires to obtain, not always successfully, in implicit methods. The situations in which these features are particularly desirable are those in which the equations have multiple significant scales in either space or time; this happens for example in combustion problems, in some problems involving two phase flow, and in problems where both boundary layers and shocks play a significant role (see, e.g., [3]).

It may be interesting to compare the present method with the random vortex method [2] where a random choice is also an essential feature. In both methods the random choice feature is used to control the numerical dissipation. However, in the random vortex method the random choice is used to represent the real viscosity as well as control the numerical viscosity. This fact makes it imperative that the random numbers picked at the various spatial locations at a fixed time be independent. In the present method this constraint does not apply and, in fact, it is essential that it be flouted.

Finally, it should be stressed that a certain randomness is a property of many real flows, and thus a method which exhibits randomness is not necessarily less desirable than a method which yields fully predictable answers.

Note. The programs used to obtain the results above are available from the author.

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